

NON-NORMAL AFFINE MONOIDS

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ABSTRACT. We give a geometric description of the set of holes in a non-normal affine monoid Q and show how this relates to various properties of Q like local normality and Serre's conditions (R_1) and (S_2) . A combinatorial upper bound for the depth the monoid algebra $k[Q]$ is obtained. Moreover, we show the vanishing and non-vanishing of certain graded components of the local cohomology of $k[Q]$. We then apply our results to simplicial and to seminormal affine monoids. Finally, we prove a special case of the Eisenbud-Goto conjecture.

1. INTRODUCTION

Let Q be an affine monoid, i.e. a finitely generated submonoid of \mathbb{Z}^N for some $N \in \mathbb{N}$. Further, let \overline{Q} denote the normalization of Q . In this paper, we give a geometric description of the set of “holes” $\overline{Q} \setminus Q$ in Q and relate it to several properties of Q . Our main result in this direction is the following.

Theorem (Theorem 3.8). *Let Q be an affine monoid. There exists a (non-disjoint) decomposition*

$$(1) \quad \overline{Q} \setminus Q = \bigcup_{i=1}^l (q_i + \mathbb{Z}F_i) \cap \mathbb{R}_+ Q$$

with $q_i \in \overline{Q}$ and faces F_i of Q . If the union is irredundant (meaning that no $q_i + \mathbb{Z}F_i$ can be omitted), then the decomposition is unique.

We call a set $q_i + \mathbb{Z}F_i$ appearing in (1) an j -*dimensional family of holes, where j is the *dimension of F (See Section 2 for the definition of the *dimension). There is an algebraic interpretation of the sets appearing in (1). Let k be a field and $k[Q]$ be the monoid algebra of Q . Then the faces appearing in (1) correspond to the associated primes of the quotient $k[\overline{Q}]/k[Q]$. The same face may appear several times in (1), in fact, the number of times a face appears equals the multiplicity of the corresponding prime.

In [1, Prop. 2.35] a different decomposition of the holes is considered. It is shown there that one can always find a decomposition of $\overline{Q} \setminus Q$ into a disjoint union of translates of faces of Q :

$$(2) \quad \overline{Q} \setminus Q = \bigcup_{i=1}^l q_i + F_i$$

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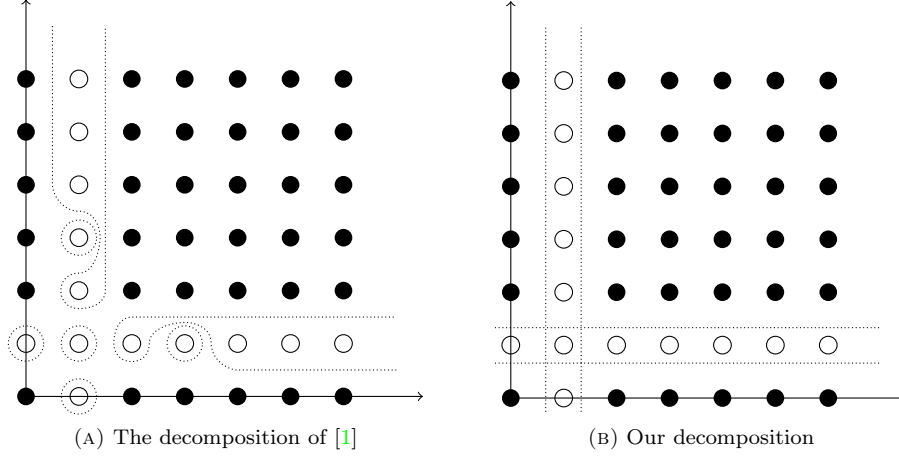


FIGURE 1. Different decomposition of the holes of a 2-dimensional affine semigroup

In fact, this statement and its proof have been the motivation for proving [Theorem 3.8](#). [Figure 1](#) shows an example of both kinds of decomposition. The decomposition given in (2) is disjoint, but far from being unique. On the other hand, our decomposition is (in general) not disjoint, but it is unique. Moreover, it behaves nicely under localization, see [Proposition 3.10](#). Several ring-theoretic properties of the monoid algebra $k[Q]$ can be described in terms of the families of holes:

Theorem ([Theorem 4.3](#)). *Let Q be an affine monoid of \ast -dimension d . The following holds:*

- *If $d \geq 2$, then $\ast\text{depth } k[Q] = 1$ if and only if there is a 0- \ast -dimensional family of holes.*
- *Q is locally normal if and only if there is no family of holes of positive \ast -dimension.*
- *$k[Q]$ satisfies Serre's condition (R_1) if and only if there is no family of holes of \ast -dimension $d - 1$.*
- *$k[Q]$ satisfies Serre's condition (S_2) if and only if every family of holes has \ast -dimension $d - 1$.*

Note that this implies that if Q is locally normal (very ample), but not normal, then $\ast\text{depth } k[Q] = 1$. We generalize the first item of the preceding theorem to obtain an upper bound on the \ast -depth of $k[Q]$:

Theorem ([Theorem 4.4](#)). *If Q has an i - \ast -dimensional family of holes, then the \ast -depth of $k[Q]$ is at most $i + 1$.*

This theorem states that a non-normal affine monoid with “few” holes has a low \ast -depth. This is somewhat counterintuitive, because Hochster's Theorem states that the absence of holes, i.e. normality, implies maximal \ast -depth. In small examples, it is often not difficult to determine the bound given by this theorem geometrically. This can be easier than to compute the actual \ast -depth algebraically. In general, the \ast -depth may be strictly smaller than the bound given by [Theorem 4.4](#). However,

in [Proposition 4.5](#), [Proposition 6.2](#) and [Proposition 6.11](#) we identify some special cases where equality holds. See [Example 4.7](#) for an application.

In the second part of this paper, we consider the local cohomology $H_{\mathfrak{m}}^i(k[Q])$ at the maximal graded ideal \mathfrak{m} . Recall that for an element $q \in -\text{int } Q$, it holds that $H_{\mathfrak{m}}^d(k[Q])_q = k$ and $H_{\mathfrak{m}}^i(k[Q])_q = 0$ for all other i . So this part of $H_{\mathfrak{m}}^i(k[Q])$ is well understood. It turns out that the remaining graded components are closely related to the families of holes of Q .

Theorem ([Theorem 5.3](#)). *Consider $q \in \mathbb{Z}Q$ such that $q \notin -\text{int } Q$. If $H_{\mathfrak{m}}^{i+1}(k[Q])_q \neq 0$ for some i , then q is contained in a family of holes of $^*\text{dimension}$ at least i .*

There is also a partial converse of the preceding result.

Corollary ([Corollary 5.5](#)). *Every i - $^*\text{dimensional}$ family of holes contains an element $q \in \mathbb{Z}Q$, such that $H_{\mathfrak{m}}^{i+1}(k[Q])_q = k$. If $i > 0$, then there are in fact infinitely many such elements (even up to units).*

Note that this result extends [Theorem 4.4](#). We will also give restrictions on certain infinite parts of the local cohomology of $k[Q]$ in [Lemma 5.6](#). See [Example 5.7](#) for an example of the geometric meaning of these results.

In [Section 6](#), we apply our results to simplicial and seminormal affine monoids. For simplicial affine monoids, the characterization of the Cohen-Macaulay property of [\[6\]](#) is extended to the non-positive case. For seminormal affine monoids, we give a new proof of the cohomological characterization of seminormality of [\[3\]](#). While our proof is not actually simpler than the original one, we believe that it offers a new, more geometric perspective. Moreover, we extend this and some other results of [\[3\]](#) to the non-positive case. Further, a simplicial seminormal affine monoid whose $^*\text{depth}$ depends on the characteristic of the field is constructed.

In the last section, some additional results are listed. We give a direct proof that out criterion for Serre's condition (S_2) is equivalent to the one given in [\[11, 20\]](#). Further, an interpretation of greatest size of a family of holes is given. Finally, we give a bound on the Castelnuovo-Mumford regularity of seminormal homogeneous affine monoids and prove a special case of the Eisenbud-Goto conjecture ([Theorem 7.5](#)).

2. PRELIMINARIES AND NOTATION

An affine monoid Q is a finitely generated submonoid of the additive monoid \mathbb{Z}^N for an $N \in \mathbb{N}$. Equivalently, Q is a commutative, finitely generated, cancellative and torsionfree monoid. For general information about affine monoids see [\[1\]](#) or [\[13\]](#). We denote the group generated by Q by $\mathbb{Z}Q$, the convex cone generated by Q by $\mathbb{R}_+Q \subseteq \mathbb{R}^N$ and the normalization by $\overline{Q} = \mathbb{Z}Q \cap \mathbb{R}_+Q$. Recall that an element $q \in \mathbb{Z}Q$ is contained in \overline{Q} if and only if a multiple of q lies in Q . A *face* $F \subseteq Q$ of Q is a subset such that for $a, b \in Q$ the following holds:

$$a + b \in F \iff a, b \in F$$

A *unit* is an element in $u \in Q$, such that $-u \in Q$. The set of units forms a face F_0 that is contained in every other face of Q . We call Q *positive* if 0 is the only unit in Q . For every element $q \in Q$, there exists a unique minimal face F containing q . We say that q is an *interior point* of F and write $\text{int } F$ for the set of interior points of F . Note that by definition $0 \in \text{int } F_0$. The dimension of a face F is the rank of the free abelian group $\mathbb{Z}F$ generated by F . Since we are working with not-necessarily positive affine monoids, it is more convenient to consider a normalized version of

the dimension. So we define the **dimension* as $^*\dim Q := \dim Q - ^*\dim F_0$, and $^*\dim F := \dim F - \dim F_0$ for every face F of Q . For a field k , we write $k[Q]$ for the monoid algebra of Q . Further, for an element $q \in Q$, we write $\mathbf{x}^q \in k[Q]$ for the corresponding monomial. For a face F we define $\mathfrak{p}_F \subseteq k[Q]$ to be the vector space generated by those monomials \mathbf{x}^q such that $q \in Q \setminus F$. Then \mathfrak{p}_F is a monomial prime ideal of $k[Q]$ and all monomial prime ideals are of this type. Moreover, $k[Q]/\mathfrak{p}_F \cong k[F]$. $k[Q]$ carries a natural $\mathbb{Z}Q$ -grading. With respect to this grading, the homogeneous ideals of $k[Q]$ are exactly the monomial ideals. Thus the ideal \mathfrak{p}_{F_0} associated to the minimal face is the unique maximal graded ideal of $k[Q]$. We will sometimes write \mathfrak{m} for this ideal. Its height equals the maximal length on an descending chain of faces of Q , so $(k[Q], \mathfrak{m})$ is a **local ring* of **dimension* $^*\dim Q$. More general, the height of \mathfrak{p}_F equals $^*\dim F$ for every face F . The **depth* of $k[Q]$ is the maximal length of a regular sequence in \mathfrak{m} . Equivalently, it is the depth of the (inhomogeneous) localization $k[Q]_{\mathfrak{m}}$. For a face F of Q , we denote by

$$Q_F := \{q - f \mid q \in Q, f \in F\}$$

the *localization* of Q at F . It holds that $k[Q_F] = k[Q]_{(\mathfrak{p}_F)}$, where the later is the homogeneous localization of $k[Q]$ at \mathfrak{p}_F . Further, it holds that $(\overline{Q})_F = \overline{(Q_F)}$, i.e. localization and normalization commutes. Note that localizations are almost never positive.

A set M is called a *Q-module* if there is an operation $Q \times M \rightarrow M$ (additively written) of Q on M , such that $(q + p) + m = q + (p + m)$ and $0 + m = m$ for $q, p \in Q, m \in M$. If M is a Q -module, then the vector space $k\{M\}$ with basis given by the elements of M is naturally a $k[Q]$ -module. If $M \subseteq \mathbb{Z}Q$, then we define the localization $M_F := \{m - f \mid m \in M, f \in F\}$ of M at a face F . One may also consider the localization for general modules, but we only need this special case. Note that if $U \subseteq M \subseteq \mathbb{Z}Q$ are modules, then $k\{M_F\}/k\{U_F\} = k\{M\}/k\{U\}_{(\mathfrak{p}_F)}$.

For a graded $k[Q]$ -module N (in the algebraic sense), the *support* of N , $\text{supp } N$, is defined to be the set of those $q \in \mathbb{Z}Q$, such that there exists an element of degree q in N . If M is a Q -module and $U \subseteq M$ a submodule, then $\text{supp } k\{M\}/k\{U\} = M \setminus U$.

Definition 2.1. Let Q be an affine monoid. We call Q *locally normal* if every localization Q_F at a face $F \neq F_0$ is normal.

Since localizations of normal affine monoids are again normal, it is enough to consider faces of **dimension* 1. For a polytope $P \subset \mathbb{R}^{N-1}$ with integer vertices, one often considers the polytopal affine monoid $Q_P \subseteq \mathbb{Z}^N$ generated by the set $\{(p, 1) \mid p \in P \cap \mathbb{Z}^{N-1}\}$. In this case, Q_P is locally normal if and only if the polytope P is *very ample*. To see this, note that the localization of Q_P at a vertex $(v, 1)$ splits into a direct sum of the *corner cone* on v and a copy of \mathbb{Z} . Thus, Q_P is locally normal if and only if all the corner cones of P are normal, which is the definition of very ampleness. It is known that P is very ample if and only if there are only finitely many holes in Q_P . The corresponding statement in the general case is the following:

Proposition 2.2. *An affine monoid is locally normal if and only if*

$$\text{rank}_{k[F_0]} k[\overline{Q}]/k[Q] < \infty$$

for any field k . This rank does not depend on the field.

Recall that the face F_0 consists of the units of Q , so this lemma amounts to saying that there are only finitely many holes up to units. Every graded module over $k[F_0]$ is free (cf. [7, Theorem 1.1.4]), so the rank is well-defined. We will not use this lemma in the sequel, so we omit the proof.

3. THE MODULE OF HOLES

In this section, we describe the structure of the set of holes $\overline{Q} \setminus Q$. Following an idea from [1, p. 139], we consider a more general situation. Let M be a finitely generated Q -submodule of $\mathbb{Z}Q$ and let $U \subseteq M$ be a submodule of M . We are interested in the structure of the difference $M \setminus U$. Clearly, in the case $M = \overline{Q}$ and $U = Q$ this corresponds to the holes $\overline{Q} \setminus Q$. While for our purpose it would actually suffice to consider this case, we believe that the additional generality makes the exposition more clear. Another case of potential interest is $N = Q$ and $U \subseteq Q$ a submodule. This corresponds to a monomial ideal in $k[Q]$. As noted above, the set $M \setminus U$ can be encoded as the support of the quotient $k\{M\}/k\{U\}$. The following simple observation is the key idea: Consider an $m \in M \setminus U$ and a $q \in Q$. Let \mathbf{x}^m and \mathbf{x}^q denote the corresponding monomials in $k\{M\}/k\{U\}$ resp. in $k[Q]$. Then

$$q + m \in U \iff \mathbf{x}^q \mathbf{x}^m = 0.$$

Now let F be a face of Q . It holds that $m \in M_F$, because $M \subseteq M_F$. However, $m \in U_F$ if and only if \mathbf{x}^m goes to zero in the localization of the quotient $k\{M\}/k\{U\}$ at \mathfrak{p}_F . This is the case if and only if the annihilator of \mathbf{x}^m is not contained in \mathfrak{p}_F , i.e. if there is a $q \in F$ such that $q + m \in U$. Consider the case that $m \notin U_F$ and F is maximal with this property. By this we mean that $m \in U_G$ for all faces $G \supsetneq F$. Because $\mathfrak{p}_G \subseteq \mathfrak{p}_F$, this is equivalent to \mathfrak{p}_F being a minimal prime over the annihilator of \mathbf{x}^m . We summarize what we have proven:

Lemma 3.1. *Let F be a face of Q , $m \in M \setminus U$ and \mathbf{x}^m be the corresponding monomial in $k\{M\}/k\{U\}$. Then $F + m \subseteq M \setminus U$ if and only if \mathfrak{p}_F contains the annihilator of \mathbf{x}^m . Moreover, F is a maximal face with this property if and only if \mathfrak{p}_F is a minimal prime of the annihilator of \mathbf{x}^m .*

In view of our objective to find an irredundant decomposition of the set $M \setminus U$, it seems natural to take the largest possible pieces. Therefore, we consider the family of sets

$$\mathcal{F}(M) = \{ \mathbb{Z}F + m \subseteq \mathbb{Z}Q \mid m \in M \setminus U, \mathfrak{p}_F \text{ is a minimal prime over } \text{Ann } \mathbf{x}^m \}.$$

These sets will yield the desired decomposition. Note that for $m, n \in M \setminus U$ with $m - n \in \mathbb{Z}F$, it holds that \mathfrak{p}_F is a minimal prime over $\text{Ann } \mathbf{x}^m$ if and only if it is minimal over $\text{Ann } \mathbf{x}^n$. So we are free in choosing representatives of the sets in $\mathcal{F}(M)$.

We first show that their union comprises all of $M \setminus U$:

Lemma 3.2. *It holds that*

$$(3) \quad M \setminus U = \bigcup_{S \in \mathcal{F}(M)} S \cap M$$

Proof. Every monomial $\mathbf{x}^m \in k\{M\}/k\{U\}$ has at least one minimal prime over its annihilator, so the left-hand side of (3) is clearly contained in the right-hand side. On the other hand, consider $n \in \mathbb{Z}F + m \cap M$ for $\mathbb{Z}F + m \in \mathcal{F}(M)$. There exist

f_1, f_2 in F such that $f_1 + m = f_2 + n$. It follows that $n \notin U$, because $F + m \subseteq M \setminus U$ by [Lemma 3.1](#), and hence $n \in M \setminus U$. \square

Next, we consider the behaviour of $\mathcal{F}(M)$ under localization.

Lemma 3.3. *Let $F \subseteq G$ be faces of Q and let $m \in \mathbb{Z}Q$. Then $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if $\mathbb{Z}G + m \in \mathcal{F}(M_F)$.*

Proof. If $m \in M_F \setminus U_F$, then there exists an $f \in F$ such that $f + m \in M \setminus U$ and, since $F \subseteq G$, it holds that $\mathbb{Z}G + m = \mathbb{Z}G + f + m$. So we can assume that $m \in M \setminus U$. In this case, $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if \mathfrak{p}_G is minimal over the annihilator of $\mathbf{x}^m \in k\{M\}/k\{U\}$. But this property is preserved under localization with $\mathfrak{p}_F \supseteq \mathfrak{p}_G$. Hence the claim follows. \square

Using the preceding lemma, we prove the finiteness of our decomposition:

Lemma 3.4. *For every face F of Q , the number of sets of the form $\mathbb{Z}F + m \in \mathcal{F}(M)$ equals the multiplicity of \mathfrak{p}_F on $k\{M\}/k\{U\}$. In particular, $\mathcal{F}(M)$ is finite. Moreover, a face F appears in $\mathcal{F}(M)$ if and only if \mathfrak{p}_F is an associated prime of $k\{M\}/k\{U\}$.*

Proof. The multiplicity of \mathfrak{p}_F in $k\{M\}/k\{U\}$ can be defined as the length of the module

$$N := H_{\mathfrak{p}_F}^0(k\{M_F\}/k\{U_F\}) = \text{span}_k \{ \mathbf{x}^m \in k\{M_F\}/k\{U_F\} \mid \mathfrak{p}_F^n \mathbf{x}^m = 0 \text{ for } n \gg 0 \},$$

see [4, p. 102]. Note that the multiplicity is invariant under localization at F . By [Lemma 3.3](#), the same holds for the number of sets of the form $\mathbb{Z}F + m \in \mathcal{F}(M)$. So we may assume that F is the minimal face of Q .

Since N is a module of finite length (in the graded sense), we may consider a composition series of N , i.e. a filtration $0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots \subsetneq N_r = N$ of graded modules, such that each quotient N_i/N_{i-1} is a simple graded module.

We claim that $\text{supp } N$ is the union of the sets $\mathbb{Z}F + m \in \mathcal{F}(M)$ (for our fixed F). Indeed, $m \in M \setminus U$ is contained in $\text{supp } N$ if and only if the annihilator of \mathbf{x}^m is \mathfrak{p}_F -primary. Since \mathfrak{p}_F is the maximal graded ideal of $k[Q]$, this is equivalent to saying that \mathfrak{p}_F is a minimal prime over the annihilator of \mathbf{x}^m . Now, for every $\mathbb{Z}F + m \in \mathcal{F}(M)$, there exists an i with $m \in \text{supp } N_i \setminus \text{supp } N_{i-1}$. But this implies that $\mathbb{Z}F + m = \text{supp } N_i \setminus \text{supp } N_{i-1}$, because N_i/N_{i-1} is simple. Thus, the number of sets of the form $\mathbb{Z}F + m$ in $\mathcal{F}(M)$ equals the length r of the composition series. \square

We now turn to proving the irredundancy and the uniqueness of (3). First, we give a variant of the well-known fact that a vector space over an infinite field cannot be written as a union of finitely many subspaces.

Lemma 3.5. *Let V be a vector space over \mathbb{Q} and $C \subseteq V$ be a convex cone (i.e. a subset such that for $v, w \in C$ and $\lambda, \mu \geq 0$ it follows $\lambda v + \mu w \in C$). If C contains a generating set of V , then it is not contained in any finite union of proper subspaces of V .*

Proof. Assume to the contrary that the cone C is contained in the union of finitely many proper subspaces V_1, \dots, V_l of V . We may further assume that none of the subspaces is contained in the union of the others and that every V_i has a non-empty intersection with C . We have certainly at least two subspaces, because C contains a generating set of V . Hence we can choose elements $x_1, x_2 \in C$, such that

$x_i \in V_i \setminus \bigcup_{j \neq i} V_j$ for $i = 1, 2$. For every $i \geq 2$ there exists at most one $\lambda \in \mathbb{Q}$ with $\lambda x_1 + x_2 \in V_i$. Indeed, if we had $\lambda x_1 + x_2 \in V_i$ and $\lambda' x_1 + x_2 \in V_i$ for two different $\lambda, \lambda' \in \mathbb{Q}$, then $(\lambda - \lambda')x_1 \in V_i$, a contradiction to our choice of x_1 . Since there are infinitely many non-negative rational numbers and it holds $\lambda x_1 + x_2 \in C$ for every such $\lambda \geq 0$, we conclude that there exists a $\lambda \in \mathbb{Q}$ such that $\lambda x_1 + x_2 \in V_1$. But now $x_2 = (\lambda x_1 + x_2) - \lambda x_1 \in V_1$, a contradiction. \square

Next we prepare a discrete analogue of the preceding lemma.

Lemma 3.6. *Let $q, p_1, \dots, p_l \in \mathbb{Z}Q$ be lattice points and let F, G_1, \dots, G_l be (not necessarily distinct) faces of Q . If $F + q$ is contained in the union $\bigcup_i \mathbb{Z}G_i + p_i$, then it is already contained in one of the sets $\mathbb{Z}G_i + p_i$.*

Note that this Lemma does not hold for arbitrary subgroups of \mathbb{Z}^N , for example $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z} + 1)$.

Proof. We may assume that $F + q$ has a non-empty intersection with every $\mathbb{Z}G_i + p_i$ for $1 \leq i \leq l$. If $F \subseteq G_i$ for any i , then $F + q \subseteq \mathbb{Z}F + q' \subseteq \mathbb{Z}G_i + q' = \mathbb{Z}G_i + p_i$ for $q' \in F + q \cap \mathbb{Z}G_i + p_i$. Thus in this case our claim holds. We will show that there exists always an i such that $F \subseteq G_i$.

So, assume to the contrary that $F \not\subseteq G_i$ for every i . As a notation, for a subset $S \subseteq \mathbb{Q}^N$, we write $\mathbb{Q}S$ for the \mathbb{Q} -subspace generated by S . Then, $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i) \subsetneq \mathbb{Q}F$ for every i . Indeed, it holds that $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i) \subseteq \mathbb{Q}F \cap \mathbb{Q}G_i \subseteq \mathbb{Q}F$. The second inclusion is strict except in the case that $\mathbb{Q}F \subseteq \mathbb{Q}G_i$. But this would imply that $F \subseteq G_i$, because $F = \mathbb{Q}F \cap Q$ and $G_i = \mathbb{Q}G_i \cap Q$. Note that this is the point where we use that we deal with faces of an affine monoid.

By Lemma 3.5, we can find an element \hat{p} in the cone generated by F that is not contained in any $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i)$. By multiplication with a positive scalar, we can assume $\hat{p} \in F$. For every non-negative integer λ , it holds that $\lambda \hat{p} + q \in F + q \subseteq \bigcup_i \mathbb{Z}G_i + p_i$. Since the union is finite, there exists an index i and two different integers $\lambda, \lambda' \in \mathbb{Z}$ such that $\lambda \hat{p} + q, \lambda' \hat{p} + q \in \mathbb{Z}G_i + p_i$. But now it follows that $(\lambda - \lambda')\hat{p} \in \mathbb{Z}F \cap \mathbb{Z}G_i$ and thus $\hat{p} \in \mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i)$, a contradiction to our choice of \hat{p} . \square

Now we are ready to prove that our decomposition is in fact irredundant and unique.

Lemma 3.7. *Consider a finite decomposition*

$$M \setminus U = \bigcup_i (\mathbb{Z}G_i + m_i) \cap M$$

of $M \setminus U$ with $m_i \in M \setminus U$ and faces G_i of Q . Then every set in $\mathcal{F}(M)$ appears in this decomposition. Thus, (3) defines the unique irredundant finite decomposition of $M \setminus U$.

Proof. Let $m \in M \setminus U$ and F a face of Q such that $\mathbb{Z}F + m \in \mathcal{F}(M)$. By Lemma 3.6, there exists an index i such that $F + m \subseteq \mathbb{Z}G_i + m_i$. In particular, $\mathbb{Z}G_i + m = \mathbb{Z}G_i + m_i$. Hence $F \subseteq \mathbb{Z}G_i$ and therefore $F \subseteq \mathbb{Z}G_i \cap Q = G_i$. On the other hand, by Lemma 3.1, F is a maximal face of Q such that $F + m \subseteq M \setminus U$. But $G_i + m \subseteq \mathbb{Z}G_i + m \cap M \subseteq M \setminus U$, so we conclude that $G_i = F$. Whence $\mathbb{Z}F + m = \mathbb{Z}G_i + m_i$. \square

We summarize what we have proven in this section in the following theorem.

Theorem 3.8. *Let Q be an affine monoid, let $M \subseteq \mathbb{Z}Q$ be a module and let $U \subseteq M$ be a submodule. Then there exists a unique irredundant finite (non-disjoint) decomposition*

$$(4) \quad M \setminus U = \bigcup_i (\mathbb{Z}F_i + m_i) \cap M.$$

The number of times F appears in (4) equals the multiplicity of \mathfrak{p}_F on $k\{M\}/k\{U\}$. In particular, a face F appears in (4) if and only if \mathfrak{p}_F is an associated prime of $k\{M\}/k\{U\}$. Geometrically, a set $\mathbb{Z}F_i + m_i$ appears in (4) if and only if F_i is a maximal face such that $F_i + m_i \subseteq M \setminus U$.

From now on, we specialize our discussion to the case $M = \overline{Q}$ and $U = Q$. Let us call a set $\mathbb{Z}F + m$ appearing in (4) a j -dimensional family of holes, where $j = \text{*dim } F$. For the ease of reference, call a face F associated to Q if \mathfrak{p}_F is an associated prime of $k[\overline{Q}]/k[Q]$. The following is immediate:

Corollary 3.9. *Q has a j -*dimensional family of holes if and only if there is a j -*dimensional associated face of Q .*

We get a description of the holes of the localization Q_F from [Lemma 3.3](#):

Proposition 3.10. *Let F be a face of Q . The families of holes of Q_F are exactly those families of holes $\mathbb{Z}G + q$ of Q which satisfy $F \subseteq G$. In particular, Q_F is normal if and only if no associated face contains F .*

4. SPECIAL CONFIGURATIONS OF HOLES

In this section, we show various ring-theoretical properties of $k[Q]$ correspond to special configurations of the holes in Q . We start by establishing the connection between the *depth and the module of holes:

Proposition 4.1. *Assume that Q is not normal. Then*

$$\text{*depth } k[Q] = 1 + \text{*depth } k[\overline{Q}]/k[Q]$$

Proof. The idea is to consider the short exact sequence

$$0 \longrightarrow k[Q] \longrightarrow k[\overline{Q}] \longrightarrow k[\overline{Q}]/k[Q] \longrightarrow 0$$

of $k[Q]$ -modules. We will prove that $k[\overline{Q}]$ is Cohen-Macaulay as a $k[Q]$ -module. This implies our claim by the depth lemma, [2, Prop. 1.2.9].

Every system of parameters $\mathbf{x} = x_1, \dots, x_d$ of $k[Q]$ is also a system of parameters of $k[\overline{Q}]$ (as a module over itself). This follows from the graded analogue of [2, Theorem A8]. Since $k[\overline{Q}]$ is Cohen-Macaulay by Hochster's Theorem [1, Thm. 6.12], \mathbf{x} is a regular sequence for $k[\overline{Q}]$. But this property depends only on the action of \mathbf{x} on $k[\overline{Q}]$, and thus \mathbf{x} is also a $k[\overline{Q}]$ -regular sequence for $k[\overline{Q}]$ as an $k[Q]$ -module. Hence $k[\overline{Q}]$ is Cohen-Macaulay as a $k[Q]$ -module. \square

Note that if $\text{*depth } k[Q] < \text{*dim } k[Q]$, then by the previous proof $\text{depth } k[\overline{Q}]_{\mathfrak{m}} > \text{dim } k[\overline{Q}]_{\mathfrak{m}}$ as a $k[Q]_{\mathfrak{m}}$ -module. Thus, by the Auslander-Buchsbaum Formula, its projective dimension is infinite. We recall the definition of Serre's conditions.

Definition 4.2. Let R be a Noetherian ring.

- (1) R satisfies Serre's condition (R_1) , if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$ of height at most 1.
- (2) R satisfies Serre's condition (S_2) , if $\operatorname{depth} R_{\mathfrak{p}} \geq \min(2, \dim R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

If R is graded, then for both properties it is enough to consider homogeneous localizations at homogeneous prime ideals. For (S_2) , this follows from [7, Corollary 1.2.4] and for (R_1) it was shown in [21, Proposition 2.1].

Theorem 4.3. *Let Q be an affine monoid of \ast -dimension d . The following holds:*

- If $d \geq 2$, then $\ast\operatorname{depth} k[Q] = 1$ if and only if there is a 0- \ast -dimensional family of holes.
- Q is locally normal if and only if there is no family of holes of positive \ast -dimension.
- $k[Q]$ satisfies Serre's condition (R_1) if and only if there is no family of holes of \ast -dimension $d - 1$.
- $k[Q]$ satisfies Serre's condition (S_2) if and only if every family of holes has \ast -dimension $d - 1$.

Proof. We start by proving the criterion for $\ast\operatorname{depth} k[Q] = 1$. By Proposition 4.1, it holds that $\ast\operatorname{depth} k[Q] = 1$ if and only if $\ast\operatorname{depth} k[\overline{Q}]/k[Q] = 0$, which in turn holds if and only if the maximal monomial ideal is an associated prime of $k[\overline{Q}]/k[Q]$. Since the maximal monomial ideal corresponds to the minimal face F_0 of Q , the claim follows from Corollary 3.9.

Next, we prove the criterion for local normality. Recall that $k[Q]$ is locally normal if its localizations at all 1- \ast -dimensional faces are normal. By Proposition 3.10, this is clearly equivalent to the statement that there are no families of holes of positive \ast -dimension.

To prove the criterion for Serre's condition (R_1) , note $k[Q]$ satisfies R_1 if and only if $k[Q_F]$ is regular for every facet F . On the other hand, by Proposition 3.10, Q_F is normal for every facet if and only if there is no $(d - 1)$ -dimensional family of holes. Every regular ring is normal, so one implication is clear. Since $\ast\dim Q_F = 1$, it remains to show that 1- \ast -dimensional normal monoid algebras are regular. But every 1- \ast -dimensional normal monoid is isomorphic to $\mathbb{Z}^m \oplus \mathbb{N}$ for some $m \in \mathbb{N}$, so the corresponding algebra is indeed regular.

Finally, we prove the criterion for Serre's condition (S_2) . By above discussion and Proposition 3.10, it holds for any face F of Q that $\ast\operatorname{depth} k[Q_F] = 1$ if and only if F is associated to Q . On the other hand, it holds that $\ast\dim k[Q_F] = 1$ if and only if F is a facet of Q . Therefore, Serre's condition (S_2) is satisfied if and only if every associated face of Q is a facet. \square

Note that the preceding theorem implies that Q is normal if and only if it satisfies (R_1) and (S_2) . Serre's Theorem ([2, Theorem 2.2.22]) states that this holds for general Noetherian rings. Further, if $\ast\dim Q \geq 2$, then Q is normal if and only if it is locally normal and $\ast\operatorname{depth} k[Q] \geq 2$. It follows from Serre's Theorem that this statement also holds for general Noetherian rings.

The first part of the preceding Theorem can be generalized to an upper bound on the \ast -depth:

Theorem 4.4. *If Q has an i - \ast -dimensional family of holes, then the \ast -depth of $k[Q]$ is at most $i + 1$.*

Proof. By [Proposition 4.1](#) we can consider the $^*\text{depth}$ of $k[\overline{Q}]/k[Q]$. By [Theorem 3.8](#) the families of holes of Q correspond to the associated primes of $k[\overline{Q}]/k[Q]$. Now the claim follows from the general fact that $^*\text{depth } k[\overline{Q}]/k[Q] \leq ^*\dim k[Q]/\mathfrak{p}$ for every $\mathfrak{p} \in \text{Ass } k[\overline{Q}]/k[Q]$, see [\[2, Prop 1.2.13\]](#). \square

We identify a special case where equality holds:

Proposition 4.5. *If $\overline{Q} \setminus Q = q + F$ for an element $q \in \overline{Q} \setminus Q$ and a face F of Q , then $^*\text{depth } k[Q] = 1 + ^*\dim F$.*

Note that it is not sufficient to require that there is only one family of holes, see [Example 5.7](#). Before we can prove [Proposition 4.5](#), we prepare another Lemma:

Lemma 4.6. *Let $q \in Q$ and F a face of Q , such that $F + q \subseteq \overline{Q} \setminus Q$. Then $\overline{F} + q \subseteq \overline{Q} \setminus Q$.*

Proof. Assume the contrary, so there exists an element $f \in \overline{F}$ such that $f + q \in Q$. We can write $f = f_1 - f_2$ with $f_1, f_2 \in F$. But $f + f_2 + q = f_1 + q \in \overline{Q} \setminus Q$ by assumption, a contradiction. \square

Proof of Proposition 4.5. By [Lemma 3.1](#), our hypothesis is equivalent to the statement that the module $k[\overline{Q}]/k[Q]$ is cyclic with annihilator \mathfrak{p}_F . Hence $k[\overline{Q}]/k[Q] \cong k[Q]/\mathfrak{p}_F = k[F]$ (The isomorphism shifts the grading). Next, recall that the $^*\text{depth}$ of a module M over a ring R equals its $^*\text{depth}$ over $R/\text{Ann } M$. Together with [Proposition 4.1](#) this yields

$$^*\text{depth}_{k[Q]} k[Q] = 1 + ^*\text{depth}_{k[F]} k[F].$$

Now we use [Lemma 4.6](#) to conclude that $\overline{Q} \setminus Q = F + q \subseteq \overline{F} + q \subseteq \overline{Q} \setminus Q$, so $F = \overline{F}$. Hence $k[F]$ is normal and the result follows from Hochster's Theorem. \square

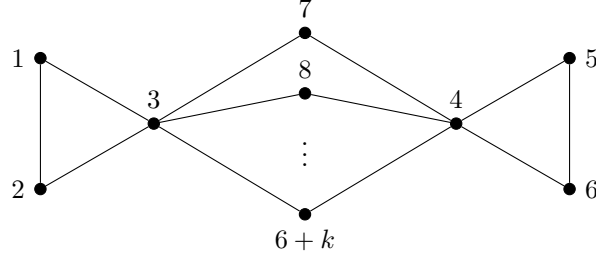
We give an example how one can effectively compute the $^*\text{depth}$ using [Proposition 4.5](#).

Example 4.7. Let G be a graph with vertex set V and edge set E . We associate an affine monoid to G , the *toric edge ring* $k[G]$, introduced in [\[15\]](#). This is the monoid algebra to the monoid generated by the vectors $\mathbf{e}_i + \mathbf{e}_j \in \mathbb{Z}^{\#V}$, where $\{i, j\}$ is an edge of G and $\mathbf{e}_i, \mathbf{e}_j$ denotes unit vectors indexed by the vertices of G .

For positive $k \in \mathbb{N}$ consider the graph G_{k+6} in [Figure 2](#). In [\[9\]](#) the $^*\text{depth}$ of the edge ring of this family of graphs is computed. We will show that these edge rings satisfy the assumption of [Proposition 4.5](#), and thus give an alternative computation of the $^*\text{depth}$.

First, it is known that $k[\overline{Q}]$ is generated as a $k[Q]$ -module by $x_1x_2x_3x_4x_5x_6$, i.e. the monomial corresponding to the vector $q \in \overline{Q} \subseteq \mathbb{R}^{k+6}$ which assigns 1 to the vertices $1, \dots, 6$. If we add one of the “middle” edges, e.g. $\{3, 8\}$, to q , then it is easy to see that the result lies in Q . On the other hand, if we add any combination of edges from the triangles to q , then the result will always be in $\overline{Q} \setminus Q$. To see this, note that the sum over the vertices of each triangle is always odd. This implies that $\overline{Q} \setminus Q = F + q$, where F is the face spanned by the six edges in the triangles. The dimension of F is 6, so by [Proposition 4.5](#) it follows that $\text{depth } k[Q] = 1 + 6 = 7$.

We generalize this computation to show that every toric edge ring can be realized as a combinatorial pure subring of a toric edge ring of $^*\text{depth}$ at most 7. The construction is as follows: To a given graph G , add two triangles on six (in total) new vertices. Then connect every vertex of G with every new vertex. Obviously, the toric edge ring of G is a combinatorial pure subring of the edge ring of this bigger

FIGURE 2. The graph G_{k+6}

graph, because G is an induced subgraph of the later. Then it is not difficult to see that the face spanned by the six edges in the triangles is associated. Its dimension is six, so the \ast depth of the ring is at most seven. In [9] it was conjectured that every toric edge ring has a \ast depth of at least seven. So we consider it as likely that the toric edge ring we constructed has a \ast depth of exactly seven.

5. LOCAL COHOMOLOGY

In this section, we consider the local cohomology of the monoid algebra $k[Q]$ with support at the maximal graded ideal $\mathfrak{m} := \mathfrak{p}_{F_0}$. We have already computed the support of the zeroth local cohomology of $k[\overline{Q}]/k[Q]$ in the proof of Lemma 3.4. Using the short exact sequence in the proof of Proposition 4.1, we obtain the following.

Proposition 5.1. *The support of $H_{\mathfrak{m}}^1(k[Q])$ equals the union of the 0- \ast dimensional families of holes.*

To extend this observation to higher dimensions we use the Ishida complex[11]. This is the complex

$$\mathcal{U}_Q : 0 \rightarrow k[Q] \rightarrow \bigoplus_{F \in \mathcal{F}_1} k[Q_F] \rightarrow \bigoplus_{F \in \mathcal{F}_2} k[Q_F] \rightarrow \cdots \rightarrow \bigoplus_{F \in \mathcal{F}_{d-1}} k[Q_F] \rightarrow k[\mathbb{Z}Q] \rightarrow 0$$

where \mathcal{F}_i denotes the set of i - \ast dimensional faces of Q . The maps are given by $\delta_i : k[Q_F] \ni \mathbf{x}^q \mapsto \sum_{G \supset F} \epsilon(F, G) \mathbf{x}^q$ via the canonical inclusion $k[Q_F] \rightarrow k[Q_G]$ for $F \subseteq G$, and $\epsilon(F, G)$ is an appropriate sign function. The modules $k[Q]$ and $k[\mathbb{Z}Q]$ sit in cohomological degree 0 resp. d . See [13, Section 13.3] for the exact definition.

Theorem 5.2 (Thm. 13.24, [13]). *The local cohomology of any $k[Q]$ -module M supported by \mathfrak{m} is the cohomology of the Ishida complex tensored with M :*

$$H_{\mathfrak{m}}^i(M) \cong H^i(M \otimes \mathcal{U}_Q)$$

Note that the last map in \mathcal{U}_Q is never surjective in degrees $q \in -\text{int } Q$, so for this q it holds that $H_{\mathfrak{m}}^d(k[Q])_q = k$ and $H_{\mathfrak{m}}^i(k[Q])_q = 0$ for $i < d$. For this reason we restrict to $q \notin -\text{int } Q$ in the following theorem.

Theorem 5.3. *Consider $q \in \mathbb{Z}Q$ such that $q \notin -\text{int } Q$. If $H_{\mathfrak{m}}^{i+1}(k[Q])_q \neq 0$ for some i , then q is contained in a family of holes of \ast dimension at least i .*

Proof. The idea is to compare the Ishida complexes for Q and its normalization \overline{Q} . Let us write $\mathcal{U}_{Q,i} := \bigoplus_{F \in \mathcal{F}_i} k[Q_F]$ for the i -th module in the complex. Consider $q \in \mathbb{Z}Q$ such that $q \notin -\text{int } Q$ and $H_{\mathfrak{m}}^i(k[Q])_q \neq 0$ for some i . It is well-known that

$H_{\mathfrak{m}}^i(k[\overline{Q}])_q = 0$ for any i , see [2, Theorem 6.3.4]. Let $x \in \mathcal{U}_{Q,i}$ be a representative of the cohomology class $H_{\mathfrak{m}}^i(k[Q])_q$. Then clearly $x \in \ker \delta_i$. Because $\mathcal{U}_{Q,i} \subseteq \mathcal{U}_{\overline{Q},i}$, x lies also in the kernel of the corresponding map in $\mathcal{U}_{\overline{Q}}$. But $H_{\mathfrak{m}}^i(k[\overline{Q}])_q = 0$, so there exists a preimage $y \in \mathcal{U}_{\overline{Q},i-1}$ of x . This element is of the form $y = \mathbf{x}^q \sum_{F \in \mathcal{F}_{i-1}} \lambda_F e_F$ where $\lambda_F \in k$ and e_F is the generator of the $k[\overline{Q}_F]$ -part of $\mathcal{U}_{\overline{Q},i-1}$.

By assumption, y is not contained in $\mathcal{U}_{Q,i-1}$. Therefore, there exists a face $F \in \mathcal{F}_{i-1}$ such that $q \in \overline{Q}_F$ but $q \notin Q_F$. Thus we can find an $f \in F$ such that $q + f \in \overline{Q} \setminus Q$ and further $q + f + F \subseteq \overline{Q} \setminus Q$. By Lemma 3.6, this implies that there is a family of holes $q_j + \mathbb{Z}F_j$ containing $q + f + F$. Hence $q \in q_j + \mathbb{Z}F_j$ and $\dim F_j \geq \dim F = i - 1$. \square

In general, the local cohomology of $k[Q]$ can be quite complicated. However, we can single out some parts that admit a simple description. First, we further evaluate the description of the local cohomology given by the Ishida complex. Following [13, Section 12.2], we construct a polytope \mathcal{P} by intersecting the cone \mathbb{R}_+Q with a suitable affine subspace perpendicular to the lineality space $\mathbb{R}F_0$. Then the face lattice of \mathcal{P} is isomorphic to the face lattice of Q . For $q \in \mathbb{Z}Q$ we set $\nabla(q) := \{F \mid q \in Q_F\}$ of faces of Q . This set is a *cocomplex*, i.e. it is closed under going up in the face lattice of Q . Turning the poset of faces of $\nabla(q)$ upside down, we get a subcomplex $\nabla(q)^\vee$ of the face lattice of the polar polytope \mathcal{P}^\vee . Because $\nabla(q)$ corresponds to the part of \mathcal{U}_Q in degree q , we can reinterpret this part as an (augmented) polyhedral chain complex for $\nabla(q)^\vee$, while reversing the cohomological degrees. So the reduced homology of the polyhedral cell complex $\nabla(q)^\vee$ gives us the local cohomology of $k[Q]$ ([13, p. 258]):

$$(5) \quad H_{\mathfrak{m}}^i(k[Q])_q = \tilde{H}_{d-i-1}(\nabla(q)^\vee, k)$$

Here, $d = \dim Q = \dim \mathcal{P} + 1$. Using this formula, we can explicitly compute a part of the local cohomology of $k[Q]$:

Proposition 5.4. *Let Q be an affine monoid and let $\mathbb{Z}F + p$ be an i -*dimensional family of holes. Let $q \in \mathbb{Z}F + p$ an element that lies beyond every facet G not containing F . By this we mean that $\sigma_G(q) < 0$, where σ_G is the supporting linear form of G . Then*

$$H_{\mathfrak{m}}^j(k[Q])_q = \begin{cases} k & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove that $\nabla(q) = \{G \mid G \supsetneq F\}$. Thus $\nabla(q)^\vee$ is the boundary complex of the face \bar{F} corresponding to F in the polar polytope \mathcal{P}^\vee . This is a sphere of dimension $\dim \bar{F} - 1 = \dim \mathcal{P} - 1 - (\dim F - 1) - 1 = d - 2 - i$. So by (5) it follows

$$H_{\mathfrak{m}}^j(k[Q])_q = \tilde{H}_{d-1-j}(S^{d-2-i}) = \begin{cases} k & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $\nabla(q)$, we first consider a face G that does not contain F . For such a G we can find a facet $G' \supset G$ that does not contain F . By our assumption, q lies beyond G' and hence $q \notin Q_{G'}$. Thus $q \notin Q_G$ and therefore $G \notin \nabla(q)$. Next, by our choice of q , it holds that $q \in \overline{Q}_F \setminus Q_F$. In particular $F \notin \nabla(q)$. Moreover, $q \in \overline{Q}_G$ for every $G \supset F$, because $\overline{Q}_G \supset \overline{Q}_F$. It remains to show that $q \in Q_G$ for every $G \supsetneq F$. So assume for the contrary that $q \in \overline{Q}_G \setminus Q_G$ for such a G . We can add an

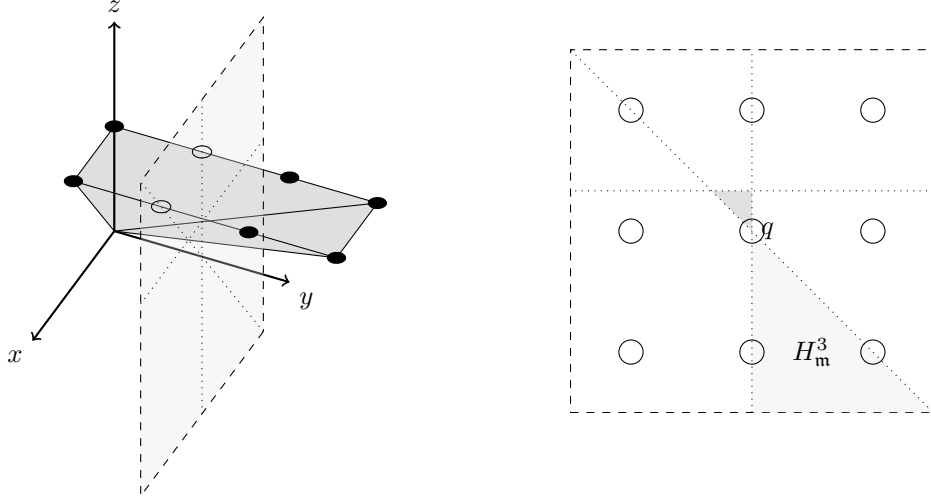


FIGURE 3. The example of Trung and Hoa

element f from the far interior of F to get $q + f \in \overline{Q}_G \setminus Q_G \cap \overline{Q}$. But this implies $G + q + f \subseteq \overline{Q} \setminus Q$, which contradicts our choice $q \in \mathbb{Z}F + p$, by [Theorem 3.8](#). \square

Note that this gives another proof of [Theorem 4.4](#).

Corollary 5.5. *Every i -*dimensional family of holes contains an element $q \in \mathbb{Z}Q$, such that $H_{\mathfrak{m}}^{i+1}(k[Q])_q = k$. If $i > 0$, then there are in fact infinitely many such elements (even up to units).*

Proof. Let $\mathbb{Z}F + p$ be a family of holes and let $q' \in \text{int } F$. For every facet $G \not\supseteq F$ it holds that $\sigma_G(q') > 0$. Hence, $p - mq'$ satisfies the hypothesis of [Proposition 5.4](#) for every sufficiently large $m \in \mathbb{N}$. This yields infinitely many non-vanishing graded components of $H_{\mathfrak{m}}^{i+1}(k[Q])$. If $i > 0$, then $F \neq F_0$ and thus $q' \notin F_0$. So these components are $k[F_0]$ -linearly independent. \square

For later use, we give a criterion for the vanishing of certain parts of the local cohomology.

Lemma 5.6. *Let q_1, q_2, \dots be a sequence of elements in $\mathbb{Z}Q$ such that $H_{\mathfrak{m}}^i(k[Q])_{q_j} \cong H_{\mathfrak{m}}^i(k[Q])_{q_1}$ for every j . Assume further that there is a facet F of Q such that $\sigma_F(q_j) < \sigma_F(q_{j+1})$ for every j . Then $H_{\mathfrak{m}}^i(k[Q])_{q_j} = 0$ for every j .*

Proof. Assume to the contrary that $H_{\mathfrak{m}}^i(k[Q])_{q_j} \neq 0$. Consider the submodules $\mathcal{H}_l \subseteq H_{\mathfrak{m}}^i(k[Q])$ generated by $H_{\mathfrak{m}}^i(k[Q])_{q_j}, j \geq l$. Clearly $\mathcal{H}_{l+1} \subseteq \mathcal{H}_l$. By our hypothesis, $\sigma_F(q_l) < \sigma_F(q_j)$ for every $j > l$. Therefore q_l is not contained in the Q -submodule of $\mathbb{Z}Q$ generated by the q_j for $j > l$. This implies that $\mathcal{H}_{l+1} \subsetneq \mathcal{H}_l$, so we get an infinite descending chain of submodules. This contradicts the fact that $H_{\mathfrak{m}}^i(k[Q])$ is Artinian, cf. [\[12\]](#)¹. \square

We give an example to demonstrate the geometric meaning of the results in this section.

¹The result is stated for the local case only, but the proof works also in the graded case.

Example 5.7. Consider the affine monoid $Q \subseteq \mathbb{Z}^3$ generated by $(0, 0, 1)$, $(1, 0, 1)$, $(0, 2, 1)$, $(1, 2, 1)$, $(0, 3, 1)$ and $(1, 3, 1)$. It is shown in the left part of [Figure 3](#). The holes of Q form a “wall” parallel to the xz -plane. Thus Q satisfies (S_2) by [Theorem 4.3](#). Moreover, the local cohomology of $k[Q]$ can only appear in the degrees of this wall. The right part of [Figure 3](#) shows this wall and the intersections with the facet defining hyperplanes. In each region, $\nabla(\cdot)$ and thus $H_m^i(k[Q])$ is constant. In the shaded unbounded region pointing downwards, we have $H_m^3(k[Q]) \neq 0$ by [Proposition 5.4](#). All other unbounded regions do not support local cohomology by [Lemma 5.6](#) (just take the points on any ray reaching outwards). So the only part of the local cohomology that is not classified so far is the lattice point q in the small shaded triangle. In fact, one may compute directly that $\dim_k H_m^2(k[Q])_q = 1$. This example is taken from [\[20\]](#), it is an affine semigroup satisfying (S_2) without being Cohen-Macaulay.

6. APPLICATIONS

We apply our methods to simplicial and seminormal affine monoid.

6.1. Simplicial affine monoids. An affine monoid Q is called *simplicial* if its face lattice is isomorphic to the face lattice of a simplex. Equivalently, one can write the cone $\mathbb{R}_{\geq 0}$ as a direct product of its linearity space and a pointed cone over a polytope \mathcal{P} . Then Q is simplicial if this polytope is a simplex. Note that we do not require Q to be positive. In this, we deviate from the terminology of [\[1\]](#). A well-known result by Goto, Suzuki and Watanabe[\[6\]](#) states that if Q is simplicial, positive and satisfies Serre’s condition (S_2) , then $k[Q]$ is Cohen-Macaulay. We give a proof of this without the positivity assumption using our description of Serre’s condition (S_2) .

Proposition 6.1. *Let Q be a simplicial affine monoid. If Q satisfies Serre’s condition (S_2) , then $k[Q]$ is Cohen-Macaulay.*

We also identify another case where the upper bound on the \ast depth given in [Proposition 4.1](#) is tight.

Proposition 6.2. *Let Q be a simplicial affine monoid. If the families of holes of Q are pairwise disjoint, then the \ast depth of $k[Q]$ equals one plus the smallest \ast dimension of a family of holes.*

Both results depend on the following lemma. Recall that we defined $\nabla(q)$ to be the set of faces of Q such that $q \in Q_F$ for $q \in \mathbb{Z}Q$. The complex $\nabla(q)^\vee$ is defined by turning the face poset of $\nabla(q)$ upside down. It is a subcomplex of the polytope \mathcal{P}^\vee polar to \mathcal{P} . Since \mathcal{P} is a simplex, the same holds for \mathcal{P}^\vee . So $\nabla(q)^\vee$ is a simplicial complex whose vertices correspond to the facets of Q . A *minimal non-face* of a simplicial complex Δ is a minimal face of the ambient simplex that is not a face of Δ .

Lemma 6.3. *Let Q be a simplicial affine monoid, $q \in \mathbb{Z}Q$ and $i \geq 2$. The $(i-1)$ -dimensional minimal non-faces of $\nabla(q)^\vee$ correspond to the $(d-i)$ - \ast dimensional families of holes containing q .*

Proof. For $q \in \mathbb{Z}Q$ let $\overline{\nabla}(q)$ denote the set of faces F of Q such that $q \in \overline{Q}_F$. We claim that $\overline{\nabla}(q)$ has a unique minimal element. For this, we write \mathcal{F}_{\geq} for the set of facets F of Q such that $\sigma_F(q) \geq 0$ and we write $\mathcal{F}_{<}$ for the set of facets F such

that $\sigma_F(q) < 0$. Our candidate for the unique minimal element is the intersection G of the facets in \mathcal{F}_{\geq} . This is indeed a face because Q is simplicial. If $p \in \text{int } G$ is an interior element, then by construction $\sigma_F(p) > 0$ for all F in $\mathcal{F}_{<}$. Therefore $q + mp \in \overline{Q}$ for $m \gg 0$ and hence $q \in \overline{Q}_G$.

On the other hand, let G' be a face such that $q \in \overline{Q}_{G'}$. Then there exists an element $g \in G'$ such that $q + g \in \overline{Q}$. It follows that $\sigma_F(g) > 0$ for all $F \in \mathcal{F}_{<}$. Hence G is not contained in any facet in $\mathcal{F}_{<}$ and can therefore be written as an intersection of facets in \mathcal{F}_{\geq} . It follows that $F \subseteq G$.

We now turn the face lattices of $\overline{\nabla}(q)$ upside down. Because $\overline{\nabla}(q)$ has a unique minimal element, $\overline{\nabla}(q)^\vee$ is isomorphic to the complex of faces of a simplex. In particular, the minimal non-faces of $\overline{\nabla}(q)^\vee$ as a subcomplex of \mathcal{P}^\vee are only vertices. Now $\nabla(q)^\vee$ is a simplicial subcomplex of $\overline{\nabla}(q)^\vee$. Its minimal non-faces inside $\overline{\nabla}(q)^\vee$ correspond to the maximal faces F such that $q \in \overline{Q}_F \setminus Q_F$. These are exactly the families of holes containing q . So the minimal non-faces of $\nabla(q)^\vee$ as a subcomplex of \mathcal{P}^\vee correspond either to the families of holes containing q or they are vertices (these come from the minimal non-faces of $\overline{\nabla}(q)^\vee$ inside \mathcal{P}^\vee). \square

Proof of Proposition 6.1. By Theorem 4.3, Serre's condition (S_2) implies that all families of holes have \ast -dimension $d - 1$. So $\nabla(q)^\vee$ is a simplicial complex with only 0-dimensional minimal non-faces for every $q \in \mathbb{Z}Q$. In other words, $\nabla(q)^\vee$ is either a simplex or empty. So the only possible (reduced) homology lies in degree -1 . By (5), this amounts to saying that $H_m^i(k[Q]) = 0$ for $i < d$, so $k[Q]$ is Cohen-Macaulay. \square

Proof of Proposition 6.2. Every $q \in \mathbb{Z}Q$ is contained in at most one family of holes. Hence $\nabla(q)^\vee$ is a simplicial complex with only one minimal non-face of positive dimension. This is either a ball or a sphere. Evaluating (5) then yields the result. \square

If Q is simplicial, then by Proposition 6.1 the Cohen-Macaulayness of $k[Q]$ does not depend on the field k . So one might wonder if the same holds more generally for the \ast -depth of $k[Q]$. However, this is wrong. We construct a counterexample below in Proposition 6.13.

6.2. Seminormal affine monoids. In this subsection, we apply our results to seminormal affine monoids. This way we reprove and extend some results of [1]. Recall that an affine monoid Q is called *seminormal* if $2q, 3q \in Q$ implies $q \in Q$ for $q \in \mathbb{Z}Q$. Equivalently, for every $q \in \overline{Q} \setminus Q$, the set $\{m \in \mathbb{N} \mid mq \in Q\}$ is contained in a proper subgroup of \mathbb{Z} . First, we give a geometric characterization of seminormality that is similar in spirit to the characterizations given in [1, p. 66f].

Proposition 6.4. *Let Q be an affine monoid. Q is seminormal if and only if for every family of holes $\mathbb{Z}F + q$ it holds that $q \in \mathbb{R}F$.*

Proof. First, assume that the condition in the statement is satisfied. Consider a family of holes $\mathbb{Z}F + q$. Since $q \in \overline{Q} \setminus Q$, there exists an $m \in \mathbb{N}$ such that $mq \in Q$. By our assumption, it holds that $mq \in F$ and therefore $jmq + q \in \mathbb{Z}F + q \cap \overline{Q} \subseteq \overline{Q} \setminus Q$ for every $j \in \mathbb{N}$. It follows that either $2q \notin Q$ or $3q \notin Q$. Thus, Q is seminormal.

On the other hand, assume there is a family of holes $\mathbb{Z}F + q$ such that $q \notin \mathbb{R}F$. Then there exists an element $p \in \mathbb{Z}F + q$ such that $p \in \text{int } \overline{G}$ and $p \notin G$ for some face $G \supset F$. Thus Q is not seminormal by [1, Proposition 2.40]. \square

Corollary 6.5. *Localizations of seminormal affine monoids are again seminormal.*

Proof. This follows from [Proposition 6.4](#) and the description of the families of holes of a localization given in [Proposition 3.10](#). \square

Corollary 6.6 (Corollary 5.4, [\[3\]](#)). *Let Q be a seminormal positive affine monoid of dimension at least 2. Then $\text{depth } k[Q] \geq 2$.*

Proof. If Q is positive, then the minimal face F_0 contains only the origin $0 \in \mathbb{Z}Q$. By [Proposition 6.4](#) every 0-dimensional family of holes would be contained in $\mathbb{R}F_0 = \{0\} \subseteq Q$, so there is no 0-dimensional family of holes. Hence the claim follows from [Theorem 4.3](#). \square

This result is not valid if one omits the requirement that Q is positive. For example, consider $Q \subseteq \mathbb{Z}^3$ defined by

$$Q = \{ (x, y, z) \in \mathbb{Z}^3 \mid x, y \geq 0, z \text{ even or } x > 0 \text{ or } y > 0 \}$$

This monoid is seminormal, has $\ast\text{dimension}$ 2 and has a 0- $\ast\text{dimensional}$ family of holes, namely the odd points in the z -axis. So it has $\ast\text{depth } k[Q] = 1$ by [Theorem 4.3](#).

Next we give a preliminary characterization of seminormality. Geometrically, we show that the graded components of the local cohomology of a seminormal affine monoid are, in a certain sense, constant on rays from the origin.

Lemma 6.7. *An affine monoid Q is seminormal if and only if it satisfies the following condition: For every $q \in \mathbb{Z}Q$ there exists a positive $m \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ and every $i \in \mathbb{N}$ it holds that $H_{\mathfrak{m}}^i(k[Q])_q \cong H_{\mathfrak{m}}^i(k[Q])_{(1+mj)q}$ (as k -vector space).*

Proof. Assume that Q is seminormal and fix an element $q \in \mathbb{Z}Q$. We will find an $m \in \mathbb{N}$ such that $\nabla(q) = \nabla((1+mj)q)$ for every $j \in \mathbb{N}$. This implies our claim by [\(5\)](#). First, note that $q \in Q_F$ implies $mq \in Q_F$ for every $m \in \mathbb{N}$. Similarly, $q \notin \overline{Q}_F$ implies $mq \notin \overline{Q}_F$ for every $m \in \mathbb{N}$. So it remains to show the following: There exists an $m \in \mathbb{N}$, such that for every face F such that $q \in \overline{Q}_F \setminus Q_F$ and every $j \in \mathbb{N}$ it holds that $(1+jm)q \in \overline{Q}_F \setminus Q_F$. For this, consider a face F of Q such that $q \in \overline{Q}_F \setminus Q_F$. By [Corollary 6.5](#), the localization Q_F is seminormal and thus the set $\{m \in \mathbb{N} \mid mq \in Q_F\}$ is contained in a proper subgroup of \mathbb{Z} . Since there are only finitely many such faces, we can choose an m in the intersection of these subgroups (for example, the product of the generators). Then $1+jm$ is not contained in any of these subgroups for every $j \in \mathbb{N}$. Whence our claim follows.

For the converse, assume that Q is not seminormal. Let $\mathbb{Z}F + q$ be a family of holes such that $q \notin \mathbb{R}F$. Then there exists a facet $G \supset F$ of Q such that $\sigma_G(q) > 0$. By [Proposition 5.4](#), we can find an $p \in \mathbb{Z}F + q$ such that $H_{\mathfrak{m}}^i(k[Q])_p \neq 0$ for $i = \ast\dim F + 1$. Now the sequence $p_j := (1+mj)p$ for $j = 0, 1, \dots$ satisfies the hypothesis of [Lemma 5.6](#), so we conclude that $H_{\mathfrak{m}}^i(k[Q])_p = 0$, a contradiction. \square

With a little more case, one can show that the m in the preceding lemma can be chosen independently of q .

Corollary 6.8 (Prop. 4.14, [\[3\]](#)). *Let Q be a seminormal affine monoid and assume that $H_{\mathfrak{m}}^i(k[Q])_q \neq 0$ for some $q \in \mathbb{Z}Q$ and $i \neq 1$. Then $\text{rank}_{k[F_0]} H_{\mathfrak{m}}^i(k[Q]) = \infty$.*

The requirement that $i \neq 1$ is essential, because $H_m^1(k[Q])$ has always finite rank. However, if Q is seminormal and positive, then the first local cohomology always vanishes by [3, Corollary 5.4] (resp. [Corollary 6.6](#)), therefore in this situation one needs no assumption on i . We extend the characterization of seminormality given in Theorem 4.7 of [3].

Theorem 6.9 (Thm. 4.7, [3]). *Let Q be an affine monoid. The following statements are equivalent:*

- (1) Q is seminormal.
- (2) $H_m^i(k[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all i .
- (3) $H_m^i(k[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all i such that Q has an $(i+1)$ -*dimensional family of holes.

Note that the third condition generalizes Theorem 4.9 in [3].

Proof. 1) \Rightarrow 2) Let Q be seminormal and let $q \in \mathbb{Z}Q$. By [Lemma 6.7](#), there exists a positive integer m such that $H_m^i(k[Q])_q \cong H_m^i(k[Q])_{(1+mj)q}$ for every i and every $j \in \mathbb{N}$. If $q \notin -\overline{Q}$, then the sequence $q_j := (1+mj)q$ satisfies the condition of [Lemma 5.6](#), so we conclude that $H_m^i(k[Q])_q = 0$.

2) \Rightarrow 3) This is obvious.

3) \Rightarrow 1) Assume that Q is not seminormal. Then, by [Proposition 6.4](#), there is a family of holes $\mathbb{Z}F + q$ of Q such that $q \notin \mathbb{R}_{\geq 0}F$. There exists a facet G containing F such that $\sigma_G(q) > 0$. By [Corollary 5.5](#), there exists an element $p \in \mathbb{Z}F + q$ such that $H_m^i(k[Q])_p \neq 0$ where $i = \dim F + 1$. The linear form σ_G is constant on $\mathbb{Z}F + q$, so $\sigma_G(p) > 0$ and hence $p \notin -\overline{Q}$. \square

There are analogues of [Proposition 6.1](#) and [Proposition 6.2](#) for the seminormal case. However, for the characterization of Cohen-Macaulayness, we need an additional assumption, namely that Q is simple. Recall that an affine monoid is called *simple* if its face lattice is isomorphic to the face lattice of a simple polytope. Equivalently, Q is simple if every localization Q_F for $F \neq F_0$ is simplicial (since vertex figures of simple polytopes are simplices, [22, Prop. 2.16]). The following result appeared in [3, Corollary 5.6] with the additional assumption that Q is positive.

Proposition 6.10. *Let Q be a simple seminormal affine monoid. If Q satisfies Serre's condition (S_2) , then $k[Q]$ is Cohen-Macaulay.*

Proposition 6.11. *Let Q be a seminormal affine monoid. If the families of holes of Q are pairwise disjoint, then the *depth of $k[Q]$ equals one plus the smallest *dimension of a family of holes.*

The proofs proceed analogously to the simplicial case, once we have shown the following fact:

Lemma 6.12. *Let Q be an seminormal affine monoid, $q \in -\overline{Q}$. Let $\overline{\nabla}(q)$ denote the set of faces of Q such that $q \in \overline{Q}_F$. Then $\overline{\nabla}(q)$ has a unique minimal element.*

Proof. The claim is a statement about \overline{Q} , so we may assume $Q = \overline{Q}$. There exists a unique face F such that $q \in -\text{int } F$. Evidently $q \in \overline{Q}_F$. We show that this F is the unique minimal element. So let G be a facet of Q that does not contain F . Then $\sigma_F(q) < 0$ and hence $q \notin \overline{Q}_G$. The same holds then for every face contained in G . It follows that every face $G \in \overline{\nabla}(q)$ is contained only in those facets that contain F . Hence, F is the unique minimal element of $\overline{\nabla}(q)$. \square

Now the proof of [Proposition 6.11](#) is completely analogous to the proof of the simplicial case. To prove [Proposition 6.11](#), note that $\overline{\nabla}(q)^\vee$ is in general a polytope. Removing vertices from the boundary complex of a polytope may indeed create homology. However, if Q is simple then $\overline{\nabla}(q)^\vee$ is a simplex, so we may again proceed analogously to the simplicial case.

We would like to point out an alternative way to derive [Proposition 6.10](#) from [Proposition 6.1](#). One needs [Proposition 6.1](#) in the non-positive setting for this argument. Assume that Q is simple, seminormal and satisfies (S_2) . Then every localization of Q is simplicial and satisfies (S_2) , hence every localization is Cohen-Macaulay. In this case, all local cohomology modules $H_{\mathfrak{m}}^i(k[Q])$ for $i < d$ have finite length (cf. [\[18, Appendix, Prop. 16\]](#)). But Q is seminormal, so [Proposition 4.14](#) in [\[3\]](#) (resp. [Corollary 6.8](#)) implies in this case that $H_{\mathfrak{m}}^i(k[Q])$ vanishes for $i < d$. In other words, $k[Q]$ is Cohen-Macaulay.

6.3. Dependence on the characteristic. We have seen in that if Q is either simplicial or seminormal and simple, then the Cohen-Macaulayness of $k[Q]$ does not depend on the field k . In contrast to this, the following construction shows that in both cases the \ast depth of $k[Q]$ can depend on the characteristic. As a notation, for $q = (q_1, \dots, q_d) \in \mathbb{N}^d$, we write $\text{supp } q = \{i \in [d] \mid q_i \neq 0\}$ and $\deg q = \sum_i q_i$. For background information on Stanley-Reisner rings, see Chapter 1 of [\[13\]](#).

Proposition 6.13. *Let Δ be an simplicial complex in the vertex set $[d]$ with Stanley-Reisner ring $k[\Delta]$. Then there exists a seminormal simplicial positive affine monoid $Q = Q(\Delta)$ of dimension d such that*

$$H_{\mathfrak{m}}^i(k[Q])_q = \begin{cases} H_{\mathfrak{m}}^{i-1}(k[\Delta])_q & \text{if } \deg q \text{ is odd and } \text{supp } q \in \Delta; \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq d-1$. If Δ is acyclic, then $\text{depth } k[Q] = \text{depth } k[\Delta] + 1$.

If Δ is the cone over the triangulated projective plane, then this yields an example of a seminormal simplicial affine monoid whose \ast depth depends on the characteristic.

Proof. We construct Q as a submonoid of \mathbb{N}^d . Let Q be the set of all elements $q \in \mathbb{N}^d$ such that either $\text{supp } q \notin \Delta$ or $\deg q$ is even. One can verify directly that this is a seminormal simplicial positive affine monoid. We identify the faces of Q with the subsets of $[d]$. The families of holes correspond then to the facets of Δ .

Next we compute the local cohomology. Let $q \in \mathbb{Z}Q$ such that $H_{\mathfrak{m}}^i(k[Q])_q \neq 0$. By [Theorem 5.3](#) it follows that $\deg q$ is odd and $\text{supp } q \in \Delta$. In particular, $q \in -\text{int } F$ for a unique face $F \in \Delta$. The set $\nabla(q)$ contains the faces containing F that are not in Δ , in other words $\nabla(q) = \{G \subseteq [d] \mid F \subseteq G, G \setminus F \notin \text{lk}_\Delta F\}$. Here $\text{lk}_\Delta F = \{G \subseteq [d] \mid G \cap F = \emptyset, F \cup G \in \Delta\}$ denotes the *link* of F in Δ . It follows that $\nabla(q)^\vee$ equals the Alexander dual of the link of F . Using [\(5\)](#), Alexander duality ([\[13, Theorem 5.6\]](#)) and Hochster's Formula ([\[8, Theorem A.7.3\]](#)), we compute

$$\begin{aligned} H_{\mathfrak{m}}^i(k[Q])_q &= \tilde{H}_{d-1-i}(\nabla(q)^\vee) \\ &= \tilde{H}^{d-|F|-(d-i)-2}(\text{lk}_\Delta F) \\ &= H_{\mathfrak{m}}^{i-1}(k[\Delta])_q. \end{aligned}$$

Here, $|F|$ denotes the number of vertices of F , so $d - |F|$ is the number of vertices of $\text{lk}_\Delta F$. In particular, it follows that $\text{depth } k[Q] \geq 1 + \text{depth } k[\Delta]$.

Now assume that Δ is acyclic. If $H_m^{i-1}(k[\Delta]) \neq 0$, then by Hochster's formula there exists a face $F \subseteq [d]$, such that $\tilde{H}^{i-|F|-2}(\text{lk}_\Delta F) \neq 0$. Because Δ is acyclic, it holds that $F \neq \emptyset$. So we can find an element $q \in -\text{int } F$ with odd degree. By above computation, it holds that $H_m^i(k[Q])_q = \tilde{H}^{i-|F|-2}(\text{lk}_\Delta F) \neq 0$ and thus $\text{depth } k[Q] \leq 1 + \text{depth } k[\Delta]$. \square

7. ADDITIONAL RESULTS

In this section, we list some additional results.

7.1. Intersection of localizations. In [16] and independently in [11] a combinatorial criterion for Serre's condition (S_2) is given. Namely, the monoid algebra ring $k[Q]$ satisfies Serre's condition (S_2) if and only if

$$(6) \quad Q = \bigcap_{F \text{ facet of } Q} Q_F.$$

We give a direct proof that this condition is equivalent to our criterion for (S_2) in [Theorem 4.3](#). It follows from the more general

Proposition 7.1. *For $0 \leq i \leq d-2$ it holds that*

$$(7) \quad \bigcap_{F \in \mathcal{F}_i} Q_F \subseteq \bigcap_{F \in \mathcal{F}_{i+1}} Q_F.$$

The inclusion is strict if and only if there exists a family of holes of i -dimension.

Proof. The inclusion is obvious, because $F \subseteq G$ implies $Q_F \subseteq Q_G$. So we need only to prove the case of equality. If the inclusion is strict, then we can choose an element q of the difference. For this q , there exists a i -dimensional face F with $q \notin Q_F$, but $q \in Q_G$ for every face $G \supsetneq F$. By [Lemma 3.1](#), this implies that \mathfrak{p}_F is a minimal prime over the annihilator of \mathbf{x}^q . Thus q is associated by [Theorem 3.8](#).

On the other hand, assume there is an i -dimensional associated face F . Then there exists a monomial $\mathbf{x}^q \in k[\overline{Q}]/k[Q]$ with annihilator \mathfrak{p}_F , in particular $q \notin Q_F$. Since $\mathfrak{p}_F \not\subseteq \mathfrak{p}_G$ for all faces G of $i+1$ -dimension, it follows from [Lemma 3.1](#) that $G + q \not\subseteq \overline{Q} \setminus Q$ and thus $q \in Q_G$ for all $(i+1)$ -dimensional faces G . Hence, q is contained in the right-hand side, but not in the left-hand side of (7). \square

Note that $Q = Q_{F_0} = \bigcap_{F \in \mathcal{F}_0} Q_F$. Therefore Q satisfies (6) if and only if all associated faces are facets. Let us add some remarks here. There is a chain of inclusions

$$(8) \quad Q \subseteq \bigcap_{F \in \mathcal{F}_1} Q_F \subseteq \bigcap_{F \in \mathcal{F}_2} Q_F \subseteq \dots \subseteq \bigcap_{F \in \mathcal{F}_{d-1}} Q_F \subseteq \overline{Q}.$$

This chain of inclusions gives rise to a similar chain on the algebra $k[Q]$ and also on the quotients modulo $k[Q]$. It yields a filtration of $k[\overline{Q}]/k[Q]$ that turns out to be the dimension filtration, as defined in [17]. It follows that if the $k[Q]$ -module $k[\overline{Q}]/k[Q]$ is sequentially Cohen-Macaulay, then *depth of $k[\overline{Q}]/k[Q]$ equals the smallest non-zero component in the filtration. In view of [Proposition 4.1](#), this means that the *depth of $k[Q]$ is one more than the smallest i -dimension of a family of holes.

7.2. The biggest family of holes. We have seen that the smallest \ast dimension of a family of holes gives an upper bound for the \ast depth. Moreover, in some cases, equality holds. As a supplement to this, we give an interpretation of the greatest \ast dimension of a family of holes.

Proposition 7.2. *Let Q be an non-normal affine monoid. The following numbers are equal:*

- (1) *The maximal \ast dimension of a family of holes of Q .*
- (2) *The minimal \ast dimension j , such that all localizations Q_F at faces of \ast dimension strictly greater than j are normal.*

If Q is homogeneous, then this number equals the degree of the difference between the Hilbert polynomials of $k[\overline{Q}]$ and $k[Q]$. If Q satisfies (R_1) , then this number equals the maximal $i < d - 1$ such that $H_{\mathfrak{m}}^{i+1}(k[Q]) \neq 0$.

See [19, Theorem 13.12] for a variant of this result (stated without proof).

Proof. The first claim is immediate from Proposition 3.10. For the statement about the Hilbert polynomials, note that the mentioned difference is just the Hilbert polynomial of $k[\overline{Q}]/k[Q]$. The statement about the local cohomology follows from Theorem 5.3 and Proposition 5.4, or more directly from the exact sequence in the proof of Proposition 4.1. \square

7.3. Regularity of seminormal affine monoids. Let Q be an affine monoid which is *homogeneous*, i.e. all elements of the Hilbert basis of Q are contained in a common affine hyperplane. In this case $k[Q]$ carries a natural \mathbb{Z} -grading such that all generators are of degree 1. The *Castelnuovo-Mumford regularity* is defined as

$$\text{reg } k[Q] := \max \{ i + j \mid H_{\mathfrak{m}}^i(k[Q])_j \neq 0 \}$$

If $q \in Q$ is an element in the interior of Q , then it is known that $H_{\mathfrak{m}}^d(k[Q])_{-q} \neq 0$. This gives us a lower bound on the regularity: If $m \in \mathbb{N}$ denotes the smallest degree of an interior element of Q , then $\text{reg } k[Q] \geq d - m$.

In the case that Q is seminormal, it was already noted in [10, Remark 5.34] that the local cohomology vanishes in positive degrees. This implies that $\text{reg } k[Q] \leq d$. We get a slightly stronger bound from Theorem 4.7 of [3] (resp. Theorem 6.9), namely the local cohomology vanishes in all non-negative degrees. Thus we get the following bound on the regularity:

Proposition 7.3. *Let Q be a homogeneous seminormal affine monoid of dimension d . Then $\text{reg } k[Q] \leq d - 1$. If Q contains an element of degree 1 in its interior, then equality holds.*

This generalizes the bound for the normal case in [19, Theorem 13.14] and the bound for the seminormal simplicial case in [14, Theorem 3.14]. A famous open conjecture in commutative algebra is the Eisenbud-Goto conjecture:

Conjecture 7.4 ([5]). *Let S be the polynomial ring with the standard grading and let $I \subseteq S$ be a homogeneous prime ideal. Then*

$$\text{reg } S/I \leq \text{mult } S/I - \text{codim } S/I$$

where mult is the multiplicity and codim is the codimension.

Theorem 7.5. *Let Q be a homogeneous affine monoid. If Q is seminormal and contains an inner point in degree 1, then Conjecture 7.4 holds for $k[Q]$.*

Proof. Let d be the dimension of Q . By the discussion above, we know that the regularity of $k[Q]$ is $d-1$. We may assume that $Q \subseteq \mathbb{Z}^d$ and $\mathbb{Z}Q = \mathbb{Z}^d$. Let \mathcal{P} be the convex hull of the elements of degree 1 of Q . So \mathcal{P} is a $(d-1)$ -dimensional convex polytope. The multiplicity of $k[Q]$ can be computed as the normalized volume of \mathcal{P} (cf. [1, Theorem 6.54]). Finally, the codimension of $k[Q]$ is $n-d$, where n is the number of generators of Q . Since every generator of Q has degree 1, n is bounded above by the number of lattice points in \mathcal{P} . So the claim follows from the following geometric proposition. \square

Proposition 7.6. *Let $\mathcal{P} \subseteq \mathbb{R}^{d-1}$ be a polytope with integral vertices and a lattice point in its interior. Let N be the number of all lattice points in \mathcal{P} . Then the normalized volume of \mathcal{P} is at least $N-1$.*

Proof. Let p be an inner lattice point in \mathcal{P} . By Carathodory's Theorem, p lies in the convex hull of d other lattice points of \mathcal{P} . Every $(d-1)$ -subset of these lattice points together with p forms a lattice simplex. Since every lattice simplex has normalized volume of at least 1, the convex hull of the d lattice points has normalized volume of at least d . Now we add the other lattice points of \mathcal{P} , one after the other. Every time, we get at least one new simplex in the convex hull, to the normalized volume increases by 1. If the number of lattice points in \mathcal{P} is N , then the normalized volume is at least $d + (N-d-1) \cdot 1 = N-1$. \square

The conclusion of Proposition 7.6 does not hold without the assumption on the existence of an inner point. For example, the triangle with vertices $(0,0)$, $(1,0)$, $(0,2)$ in the plane has four lattice points, but the normalized volume is only 2. So this approach cannot be used to prove Conjecture 7.4 for more general seminormal affine monoids.

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